

# EIGENVALUES UNDER THE RICCI FLOW OF MODEL GEOMETRIES

SONGBO HOU

**ABSTRACT.** In this paper, we study the evolving behaviors of the first eigenvalue of Laplace-Beltrami operator under the normalized Ricci flow of model geometries. In every Bianchi class, we estimate the derivative of the eigenvalue. Then we construct monotonic quantities under the Ricci flow and obtain upper and lower bounds for the eigenvalue.

## 1. INTRODUCTION

The first eigenvalue of the Laplace-Beltrami operator plays an important role in studying the geometry and topology of manifolds. Especially the estimates of upper and lower bounds of the first eigenvalue have attracted many attentions and are an interesting subject.

The classic results focus on the manifolds with fixed metrics. The research of the eigenvalue under the Ricci flow originates from [15]. In that paper, Perelman proved that the first eigenvalue of  $-\Delta + 4R$  is nondecreasing under the Ricci flow, where  $R$  is the scalar curvature of a evolving metric on a closed manifold. Using the monotonicity property, he derived that there are no nontrivial steady or expanding breathers. Later, Cao [1] considered eigenvalues of the operator  $-\Delta + \frac{R}{2}$  on manifolds with nonnegative curvature operator. He derived the evolution equation and got that the eigenvalues are nondecreasing along the Ricci flow. In [13], Li improved Cao's result without assuming nonnegative curvature operator, based on the same technique. The monotonicity of the first eigenvalues of  $-\Delta + aR$  ( $a \geq \frac{1}{4}$ ) was proved by Cao using evolution formulae [2] and by Li using entropy functionals [13]. Similar results hold under the normalized Ricci flow for the case  $a = \frac{1}{4}$  with nonpositive average scalar curvature [2]. In [3], we derived monotonicity formulae for the first eigenvalue of  $-\Delta + aR$  ( $0 < a \leq 1/2$ ) under the Ricci flow and the normalized Ricci flow. Using ODE techniques, we obtained various estimates on closed surfaces in terms of different Euler characteristic classes. Especially the first eigenvalue

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multiplied by  $e^{rt}$  is nondecreasing under the normalized Ricci flow if  $R \geq 0$ , where  $r$  is the average of  $R$ . Similar result holds for  $-\Delta + aR$  ( $a \geq \frac{1}{4}$ ) without any curvature assumption [2].

For the Laplace-Beltrami operator, Ma [14] studied monotonicity of the first eigenvalue along Ricci flow on a compact domain with Dirichlet boundary condition. Ling [10] proved the appropriate multiples of the eigenvalues are monotonic along the normalized Ricci flow on closed manifolds. In [5], the author studied the eigenvalue under the backward Ricci flow on locally homogeneous 3-manifolds, and proved in cases where the backward Ricci flow converges to a sub-Riemannian geometry after a rescaling, the eigenvalue evolves towards zero. Related references include [11, 12]

Since homogeneous geometries are the basic decomposed pieces of the geometrization of 3-manifolds, it is significant to understand the behavior of the Ricci flows in this basic case. In [6], J. Isenberg and M. Jackson studied the Ricci flows on locally homogeneous 3-manifolds and described their behaviors explicitly. Later, Knopf and McLeod [8] studied the quasi-convergence of the Ricci flows of homogeneous models. Isenberg, Jackson and Lu [7] analyzed the behaviors of the Ricci flow on locally homogenous closed 4-manifolds. Lott [13] used the concept of groupoids and solitons to interpret the longtime behaviors of the Ricci flow on locally homogeneous closed manifolds further.

The homogeneous geometries on simply connected manifolds consist of nine classes and are divided into two sets. The first set contains  $\mathbb{R}^3$ ,  $SU(2)$ ,  $SL(2, \mathbb{R})$ , Heisenberg,  $E(1, 1)$ , and  $E(2)$ . The other set contains  $H(3)$ ,  $H(2) \times \mathbb{R}^1$ , and  $SO(3) \times \mathbb{R}^1$ . The notations  $H(n)$ ,  $E(1, 1)$  and  $E(2)$  denote the group of isometries of the hyperbolic  $n$ -space, the group of isometries of the plane with flat Lorentz metric, the group of isometries of the Euclidian plane, respectively. The first set is called Bianchi classes [6]. For any given metric  $g_0$  on one of Bianchi classes, there is a Milnor frame  $\{X_1, X_2, X_3\}$  so that the metric and the Ricci tensors are diagonal and such property is preserved by the Ricci flow. Denote  $\{\theta^1, \theta^2, \theta^3\}$  the dual coframe. Then the Ricci flow takes the form

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2,$$

and is reduced to an ODE system involving three variables  $\{A(t), B(t), C(t)\}$ .

In this paper, we consider the first eigenvalue under the normalized Ricci flow on locally homogeneous closed 3-manifolds. We construct various monotonic quantities

under the Ricci flow and get estimates of the eigenvalue. Our method is to compare the components of the evolving Ricci tensor, and then control the derivative of the eigenvalue. Finally, the integration yields the results.

## 2. EVOLUTION EQUATION

In this section, we give the evolution equation for the first eigenvalue  $\lambda$ . This is just a special case ( $a = 0$ ) of Lemma 3.1 in [3]. For completeness, we provide the proof here.

**Theorem 2.1.** *Let  $(M, g(t))$  be a solution to the normalized Ricci flow on a locally homogeneous 3-manifold. Denote  $-\Delta$  the Laplace-Beltrami operator and  $\lambda(t)$  the first eigenvalue. Assume that  $u(x, t) > 0$  satisfies*

$$-\Delta u = \lambda u,$$

*with  $\int u^2(x, t) d\mu = 1$ . Then along the normalized Ricci flow, we obtain*

$$(2.1) \quad \frac{d}{dt} \lambda = \int (2R_{ij} \nabla_i u \nabla_j u) d\mu - \frac{2}{3} R \lambda.$$

*Proof.* Following calculations in [1], then we have

$$\frac{d}{dt} \lambda = \int \left( \frac{2r}{3} u \Delta u - 2u R_{ij} \nabla_i \nabla_j u \right) d\mu$$

where

$$r = \frac{\int_M R d\mu}{\int_M d\mu}$$

is the average of the scalar curvature, which equals  $R$  on locally homogeneous manifolds. Integrating by parts and using contracted Bianchi identity, we have

$$- \int 2u R_{ij} \nabla_i \nabla_j u d\mu = \int [(2u \nabla_i R_{ij}) \nabla_j u] d\mu + \int (2R_{ij} \nabla_i u \nabla_j u) d\mu,$$

and

$$\begin{aligned} \int (2\nabla_i R_{ij}) u \nabla_j u d\mu &= \int u (\nabla_j R) \nabla_j u d\mu = - \int R u \Delta u d\mu - \int R |\nabla u|^2 d\mu \\ &= \lambda R - \lambda R = 0. \end{aligned}$$

We obtain

$$\frac{d}{dt} \lambda = \int (2R_{ij} \nabla_i u \nabla_j u) d\mu - \frac{2}{3} R \lambda.$$

□

## 3. $\mathbb{R}^3$

In this class,

$$g(t) = g_0, \forall t \geq 0$$

and  $\lambda(t)$  is a constant.

#### 4. SU(2)

Given a metric  $g_0$ , we fix a Milnor frame  $\{X_i\}_1^3$  such that

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3.$$

Under the normalization  $A_0 B_0 C_0 = 1$ , the nonzero components of Ricci tensor are (refer to examples on page 171 in [4])

$$(4.1) \quad \begin{cases} R_{11} = \frac{1}{2}A[A^2 - (B - C)^2], \\ R_{22} = \frac{1}{2}B[B^2 - (A - C)^2], \\ R_{33} = \frac{1}{2}C[C^2 - (A - B)^2], \end{cases}$$

and the scalar curvature is

$$(4.2) \quad R = \frac{1}{2}[A^2 - (B - C)^2] + \frac{1}{2}[B^2 - (A - C)^2] + \frac{1}{2}[C^2 - (A - B)^2].$$

The Ricci flow equations are

$$(4.3) \quad \begin{cases} \frac{dA}{dt} = \frac{2}{3}A[-A(2A - B - C) + (B - C)^2], \\ \frac{dB}{dt} = \frac{2}{3}B[-B(2B - A - C) + (A - C)^2], \\ \frac{dC}{dt} = \frac{2}{3}C[-C(2C - A - B) + (A - B)^2]. \end{cases}$$

We assume without loss of generality that  $A_0 \geq B_0 \geq C_0$ .

**Lemma 4.1.** (Isenberg and Jackson [6]) *Let  $(A, B, C)$  be a solution to system (4.3). Then we have the following results:*

(1)  $A \geq B \geq C$  and  $A - C \leq (A_0 - C_0)e^{-2C_0^2 t}$ .

(2) *The flow converges exponentially to the fixed points  $A = B = C = 1$ , namely the round metric on the three sphere.*

In the following, we denote  $\tau$  a constant which may vary from line to line and from section to section. We get the following theorem.

**Theorem 4.1.** *Let  $\lambda(t)$  be the first eigenvalue of  $-\Delta$ . Then there is time  $\tau$  such that  $\lambda(t)e^{\int_\tau^t (\frac{2}{3}R - 2R_{33})dt}$  is nondecreasing along the normalized Ricci flow, whereas  $\lambda(t)e^{\int_\tau^t (\frac{2}{3}R - 2R_{11})dt}$  is nonincreasing. Moreover, we get*

$$e^{\frac{2(A_0 - C_0)}{C_0^2}(e^{-2C_0^2 t} - e^{-2C_0^2 \tau})} \lambda(\tau) \leq \lambda(t) \leq e^{\frac{5(C_0 - A_0)}{2C_0^2}(e^{-2C_0^2 t} - e^{-2C_0^2 \tau})}$$

for  $t \geq \tau$ .

As  $t$  goes to  $\infty$ ,  $\lambda(t)$  approaches a constant which corresponds to the eigenvalue of the round metric on the three sphere.

*Proof.* By Lemma 4.1, we have  $B - C$ ,  $A - C$  and  $A - B$  all converge exponentially to zero. Then  $R_{ij}$  approaches  $\frac{1}{2}$  and  $R$  approaches  $\frac{3}{2}$  as  $t$  goes to infinity. Using (4.1) and Lemma 4.1, we arrive at

$$\begin{aligned} R_{11} - R_{22} &= \frac{1}{2}[A^3 - A(B - C)^2] - \frac{1}{2}[B^3 - B(A - C)^2] \\ &= \frac{1}{2}(A - B)[(A + B)^2 - C^2] \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} R_{22} - R_{33} &= \frac{1}{2}[B^3 - B(A - C)^2] - \frac{1}{2}[C^3 - C(A - B)^2] \\ &= \frac{1}{2}(B - C)[(B + C)^2 - A^2] \\ &\geq 0 \end{aligned}$$

after a time  $\tau$ . Thus we obtain

$$R_{11} \geq R_{22} \geq R_{33}$$

and

$$2R_{33}\lambda - \frac{2}{3}R\lambda \leq \frac{d\lambda}{dt} \leq 2R_{11}\lambda - \frac{2}{3}R\lambda$$

if  $t \geq \tau$ .

Then  $\lambda(t)e^{\int_{\tau}^t (\frac{2}{3}R - 2R_{33})dt}$  is nondecreasing along the normalized Ricci flow, whereas  $\lambda(t)e^{\int_{\tau}^t (\frac{2}{3}R - 2R_{11})dt}$  is nonincreasing.

Since  $2C - A - B \leq 0$ , then  $C$  is nondecreasing and consequently  $C(t) \leq 1$ ,  $A(t) \geq 1$ . Furthermore, we get

$$\begin{aligned} &2R_{33} - \frac{2}{3}R \\ &= C[C^2 - (A - B)^2] - \frac{1}{3}[A^2 + B^2 + C^2 - (B - C)^2 - (A - C)^2 - (A - B)^2] \\ &\geq C[C^2 - (A - B)^2] - \frac{1}{3}(A^2 + B^2 + C^2) \\ &\geq C^3 - C(A - C)^2 - A^2 \\ &\geq C^3 - (A - C)^2 - A^3 \\ &= -(A - C)(A^2 + AC + C^2 + A - C) \\ &\geq -4(A - C) \geq -4(A_0 - C_0)e^{-2C_0^2t} \end{aligned}$$

and

$$\begin{aligned}
& 2R_{11} - \frac{2}{3}R \\
&= A[A^2 - (B - C)^2] - \frac{1}{3}[A^2 + B^2 + C^2 - (B - C)^2 - (A - C)^2 - (A - B)^2] \\
&\leq A^3 - C^2 + (A - C) \\
&\leq A^3 - C^3 + (A - C) \leq 5(A - C) = 5(A_0 - C_0)e^{-2C_0^2 t}
\end{aligned}$$

after a time  $\tau$ .

Then

$$-4(A_0 - C_0)e^{-2C_0^2 t} \leq \frac{1}{\lambda} \frac{d\lambda}{dt} \leq 5(A_0 - C_0)e^{-2C_0^2 t}$$

which implies  $\lim_{t \rightarrow \infty} \lambda(t)$  exists.

Integration for  $\tau$  to  $t$  yields

$$e^{\frac{2(A_0 - C_0)}{C_0^2}(e^{-2C_0^2 t} - e^{-2C_0^2 \tau})} \lambda(\tau) \leq \lambda(t) \leq e^{\frac{5(C_0 - A_0)}{2C_0^2}(e^{-2C_0^2 t} - e^{-2C_0^2 \tau})}.$$

□

## 5. $\text{SL}(2, \mathbb{R})$

Given a metric  $g_0$ , we fix a Milnor frame such that

$$[X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3.$$

Under the normalization  $A_0 B_0 C_0 = 1$ , the nonzero curvature components are

$$(5.1) \quad \begin{cases} R_{11} = \frac{1}{2}A[A^2 - (B - C)^2], \\ R_{22} = \frac{1}{2}B[B^2 - (A + C)^2], \\ R_{33} = \frac{1}{2}C[C^2 - (A + B)^2], \\ R = \frac{1}{2}[A^2 - (B - C)^2] + \frac{1}{2}[B^2 - (A + C)^2] + \frac{1}{2}[C^2 - (A + B)^2]. \end{cases}$$

Then the Ricci flow equations are

$$(5.2) \quad \begin{cases} \frac{dA}{dt} = \frac{2}{3}[-A^2(2A + B + C) + A(B - C)^2], \\ \frac{dB}{dt} = \frac{2}{3}[-B^2(2B + A - C) + B(A + C)^2], \\ \frac{dC}{dt} = \frac{2}{3}[-C^2(2C + A - B) + C(A + B)^2] \end{cases}$$

Without loss of generality, we may assume that  $B_0 > C_0$ .

**Lemma 5.1.** (Isenberg and Jackson [6]) *Let  $(A, B, C)$  be a solution to system (5.2). Then we have the following results:*

(1)  $B \geq C$  for all time  $t$ .

(2)  $B(t) \geq C_0 + \frac{2}{3}t$ ,  $C(t) \geq C_0 + \frac{2}{3}t$  and  $A \leq (C_0 + \frac{2}{3}t)^{-2}$ .

(3) There is time  $\tau$  such that  $A \leq B$  for all  $t \geq \tau$ ,  $B - C \leq (B_\tau - C_\tau)e^{-kt}$  and  $B \leq B_\tau + \frac{4}{3}t$ , where  $C_\tau := C(\tau)$ ,  $k := \frac{10}{3}C_\tau^2$ .

The Ricci flow evolves toward the pancake degeneracy.

**Theorem 5.1.** Let  $\lambda(t)$  be the first eigenvalue of  $-\Delta$ . Then there is time  $\tau$  such that  $\lambda(t)e^{\int_\tau^t (\frac{2}{3}R - 2R_{33})dt}$  is nondecreasing along the normalized Ricci flow, whereas  $\lambda(t)e^{\int_\tau^t (\frac{2}{3}R - 2R_{11})dt}$  is nonincreasing. Moreover, we get

$$\lambda(\tau)e^{-2(t-\tau)} \leq \lambda(t) \leq \lambda(\tau) \left( \frac{C_0 + \frac{2}{3}t}{C_0 + \frac{2}{3}\tau} \right)^3$$

for  $t \geq \tau$ .

*Proof.* First, let us compare  $R_{11}$  with  $R_{22}$ :

$$\begin{aligned} R_{22} - R_{11} &= \frac{1}{2}[B^3 - B(A^2 + C^2 + 2AC)] - \frac{1}{2}[A^3 - A(B^2 + C^2 - 2BC)] \\ &= \frac{1}{2}[(B - A)(A^2 + B^2 + 2AB - C^2) - 4ABC] \\ &= \frac{1}{2}[(B - A)(A + B + C)(A + B - C) - 4] \end{aligned}$$

since  $ABC = 1$ .

Then by Lemma 5.1, we have

$$\begin{aligned} &(B - A)(A + B + C)(A + B - C) \\ &\leq B(A + 2B)(A + B - C) \\ &= A^2B + 2AB^2 + AB(B - C) + 2B^2(B - C), \end{aligned}$$

$$A^2B \leq \left( C_0 + \frac{2}{3}t \right)^{-4} \left( B_\tau + \frac{4}{3}t \right),$$

$$2AB^2 = 2ABC \times \frac{B}{C} = \frac{2B}{C},$$

$$AB(B - C) \leq \left( C_0 + \frac{2}{3}t \right)^{-2} \left( B_\tau + \frac{4}{3}t \right) (B_\tau - C_\tau) e^{-kt},$$

and

$$2B^2(B - C) \leq 2 \left( B_\tau + \frac{4}{3}t \right)^2 (B_\tau - C_\tau) e^{-kt}$$

after a time  $\tau$ .

Noting that  $B/C$  approaches 1 and other terms shrink to zero as  $t$  goes to  $\infty$ , we get  $R_{11} > R_{22}$  with  $t \geq \tau$ .

Next we compare  $R_{11}$  with  $R_{33}$ . By Lemma 5.1, it is easy to see that

$$A(t) \geq \frac{1}{(B_\tau + \frac{4}{3}t)^2}$$

and then  $R_{11} > 0$  and  $R_{33} < 0$  if  $t$  is large enough.

Finally for  $R_{22}$  and  $R_{33}$ , we get the following identity

$$\begin{aligned} R_{22} - R_{33} &= \frac{1}{2}B[B^2 - (A + C)^2] - \frac{1}{2}C[C^2 - (A + B)^2] \\ &= \frac{1}{2}(B - C)[(B + C)^2 - A^2] \end{aligned}$$

and obtain  $R_{22} \geq R_{33}$  after a time  $\tau$ . So we have  $R_{11}(t) > R_{22}(t) \geq R_{33}(t)$  if  $t \geq \tau$ .

Then we can rewrite the evolving equation for  $\lambda$

$$\left(2R_{33} - \frac{2}{3}R\right) \lambda \leq \frac{d\lambda}{dt} \leq \left(2R_{11} - \frac{2}{3}R\right) \lambda.$$

This yields that

$$\frac{d}{dt} \left( \lambda(t) e^{\int_{\tau}^t (\frac{2}{3}R - 2R_{33})dt} \right) \geq 0$$

and

$$\frac{d}{dt} \left( \lambda(t) e^{\int_{\tau}^t (\frac{2}{3}R - 2R_{11})dt} \right) \leq 0.$$

Then  $\lambda(t) e^{\int_{\tau}^t (\frac{2}{3}R - 2R_{33})dt}$  is nondecreasing for  $t \geq \tau$  along the normalized Ricci flow, whereas  $\lambda(t) e^{\int_{\tau}^t (\frac{2}{3}R - 2R_{11})dt}$  is nonincreasing.

Moreover, the behaviors of  $A, B, C$  imply the existence of  $\tau$  such that

$$\begin{aligned} &2R_{33} - \frac{2}{3}R \\ &= C[C^2 - (A + B)^2] + \frac{1}{3}(A^2 + B^2 + C^2 + 2AC + 2AB - 2BC) \\ &= \frac{1}{3}A^2 + \frac{2}{3}AC + \frac{2}{3}AB + \frac{1}{3}(B - C)^2 - C(B + C)(B - C) \\ &\quad - 2ABC - A^2C \\ &\geq \frac{1}{3}(A^2 + AC + AB) - 2 - A^2C \\ &\geq -2 \end{aligned}$$

after  $\tau$  since  $B - C$  decays exponentially to zero and  $A \leq (C_0 + \frac{2}{3}t)^{-2}$ ,



$$\begin{aligned}
& 2R_{11} - \frac{2}{3}R \\
&= A[A^2 - (B - C)^2] + \frac{1}{3}(A^2 + B^2 + C^2 + 2AB + 2AC - 2BC) \\
&\leq \frac{1}{3}(A^2 + (B - C)^2 + 2AB + 2AC + 3A^3) \\
&\leq \frac{1}{3}(2A^2 + 2AB + 2AC) \\
&\leq \frac{1}{3}(2AB + 2AB + 2AB) \\
&= 2AB \leq \frac{2}{C} \leq \frac{2}{C_0 + \frac{2}{3}t}
\end{aligned}$$

for all  $t \geq \tau$ .

Integration from  $\tau$  to  $t$  gives

$$\lambda(\tau)e^{-2(t-\tau)} \leq \lambda(t) \leq \lambda(\tau) \left( \frac{C_0 + \frac{2}{3}t}{C_0 + \frac{2}{3}\tau} \right)^3.$$

□

## 6. HEISENBERG

In this class, given a metric  $g_0$ , there is fixed Milnor frame such that

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = 0, \quad [X_1, X_2] = 0.$$

Under the normalization  $A_0 B_0 C_0 = 1$ , the curvature components for metrics are

$$(6.1) \quad \left\{ \begin{array}{l} R_{11} = \frac{1}{2}A^3, \\ R_{22} = -\frac{1}{2}A^2B \\ R_{33} = -\frac{1}{2}A^2C, \\ R = -\frac{1}{2}A^2. \end{array} \right.$$

The Ricci flow equations are then

$$(6.2) \quad \left\{ \begin{array}{l} \frac{d}{dt}A = -\frac{4}{3}A^3, \\ \frac{d}{dt}B = \frac{2}{3}A^2B, \\ \frac{d}{dt}C = \frac{2}{3}A^2C. \end{array} \right.$$

and the solution is

$$(6.3) \quad \begin{cases} A = A_0(1 - \frac{16}{3}R_0t)^{-1/2}, \\ B = B_0(1 - \frac{16}{3}R_0t)^{1/4}, \\ C = C_0(1 - \frac{16}{3}R_0t)^{1/4}. \end{cases}$$

where  $R_0 = -\frac{1}{2}A_0^2$ .

Then Ricci flow in this class approaches the pancake degeneracy.

Assume that  $B_0 \geq C_0$ . We get the following theorem.

**Theorem 6.1.** *Let  $\lambda(t)$  be the first eigenvalue of  $-\Delta$ . Then  $\lambda(t)e^{\int_0^t (A^2B - \frac{1}{3}A^2)dt}$  is nondecreasing along the normalized Ricci flow, whereas  $\lambda(t)e^{\int_0^t -(\frac{1}{3}A^2 + A^3)dt}$  is nonincreasing. Moreover, we get*

$$\lambda(t) \geq \lambda(0) \left[ \left(1 - \frac{16}{3}R_0t\right)^{1/8} e^{-\frac{3B_0}{2}[(1 + \frac{8}{3}A_0^2t)^{1/4} - 1]} \right]$$

and

$$\lambda(t) \leq \lambda(0) \left[ \left(1 - \frac{16}{3}R_0t\right)^{1/8} e^{-\frac{A_0}{4}[(1 + \frac{8}{3}A_0^2t)^{-1/2} - 1]} \right].$$

*Proof.* By (6.1) and (6.3), we have  $R_{11} > R_{33} \geq R_{22}$ . It follows from (2.1) that

$$(6.4) \quad (\frac{1}{3}A^2 - A^2B)\lambda \leq \frac{d}{dt}\lambda \leq (\frac{1}{3}A^2 + A^3)\lambda.$$

Thus we get

$$\frac{d}{dt} \left( \lambda(t)e^{\int_0^t (A^2B - \frac{1}{3}A^2)dt} \right) \geq 0$$

and

$$\frac{d}{dt} \left( \lambda(t)e^{\int_0^t -(\frac{1}{3}A^2 + A^3)dt} \right) \leq 0.$$

Then  $\lambda(t)e^{\int_0^t (A^2B - \frac{1}{3}A^2)dt}$  is nondecreasing along the normalized Ricci flow, whereas  $\lambda(t)e^{\int_0^t -(\frac{1}{3}A^2 + A^3)dt}$  is nonincreasing.

Furthermore, we have

$$\frac{1}{3}A^2 - A^2B = \frac{A_0^2}{3(1 - \frac{16}{3}R_0t)} - \frac{A_0^2B_0}{(1 - \frac{16}{3}R_0t)^{3/4}}$$

and

$$\frac{1}{3}(A^2 + A^3) = \frac{A_0^2}{3(1 - \frac{16}{3}R_0t)} + \frac{A_0^3}{3(1 - \frac{16}{3}R_0t)^{3/2}}.$$

Integration from 0 to  $t$  yields

$$\begin{aligned}\lambda(t) &\geq \lambda(0) \left[ \left(1 - \frac{16}{3}R_0t\right)^{-\frac{A_0^2}{16R_0}} e^{\frac{3A_0^2B_0}{4R_0}[(1-\frac{16}{3}R_0t)^{1/4}-1]} \right] \\ &= \lambda(0) \left[ \left(1 - \frac{16}{3}R_0t\right)^{1/8} e^{-\frac{3B_0}{2}[(1+\frac{8}{3}A_0^2t)^{1/4}-1]} \right]\end{aligned}$$

and

$$\begin{aligned}\lambda(t) &\leq \lambda(0) \left[ \left(1 - \frac{16}{3}R_0t\right)^{-\frac{A_0^2}{16R_0}} e^{\frac{A_0^3}{8R_0}[(1-\frac{16}{3}R_0t)^{-1/2}-1]} \right] \\ &= \lambda(0) \left[ \left(1 - \frac{16}{3}R_0t\right)^{1/8} e^{-\frac{A_0}{4}[(1+\frac{8}{3}A_0^2t)^{-1/2}-1]} \right]\end{aligned}$$

□

## 7. E(1,1)

Given a metric  $g_0$ , we choose a fixed Milnor frame such that

$$[X_1, X_2] = 0, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2.$$

Under the normalization  $A_0B_0C_0 = 1$ , the nonzero curvature components of the metric are

$$(7.1) \quad \begin{cases} R_{11} = \frac{1}{2}A(A^2 - B^2), \\ R_{22} = \frac{1}{2}B(B^2 - A^2), \\ R_{33} = -\frac{1}{2}C(A + B)^2, \\ R = -\frac{1}{2}(A + B)^2. \end{cases}$$

The Ricci flow equations are

$$(7.2) \quad \begin{cases} \frac{dA}{dt} = \frac{2}{3}[-2A^3 - AB(A - B)], \\ \frac{dB}{dt} = \frac{2}{3}[-2B^3 + AB(A - B)] \\ \frac{dC}{dt} = \frac{2}{3}C(A + B)^2. \end{cases}$$

Without loss of generality, we may assume that  $A_0 \geq B_0$ .

**Lemma 7.1.** (Isenberg and Jackson [6]) *Let  $(A, B, C)$  be a solution to system (7.2). Then we have the following results:*

(1)

$$B_0(1 + \frac{8}{3}A_0^2t)^{-1/2} \leq B \leq A \leq A_0(1 + \frac{8}{3}A_0^2t)^{-1/2}$$

and

$$C_0 + \frac{4}{3}t \leq C \leq C_0 + \frac{8}{3} \left( \frac{A_0}{B_0} \right) t.$$

(2)

$$(A_0 - B_0)(1 + \frac{8}{3}A_0^2t)^{-2} \leq A - B \leq (A_0 - B_0)(1 + \frac{8}{3}B_0^2t)^{-2}$$

and

$$(A_0 + B_0)(1 + \frac{8}{3}A_0^2t)^{-1} \leq A + B \leq (A_0 + B_0)(1 + \frac{8}{3}B_0^2t)^{-1}.$$

Then Ricci flow in this class approaches the cigar degeneracy.

**Theorem 7.1.** *Let  $\lambda(t)$  be the first eigenvalue of  $-\Delta$ . Then there is time  $\tau$  such that  $\lambda(t)e^{\int_\tau^t -[\frac{1}{3}(A+B)^2 - C(A+B)^2]dt}$  is nondecreasing along the normalized Ricci flow, whereas  $\lambda(t)e^{\int_\tau^t -[\frac{1}{3}(A+B)^2 + A(A^2 - B^2)]dt}$  is nonincreasing. Moreover, we get*

$$\lambda(\tau)\left(\frac{t}{\tau}\right)^{-c_2} \leq \lambda(t) \leq \lambda(\tau)e^{c_1(\frac{1}{\tau} - \frac{1}{t})}$$

for  $t \geq \tau$ , where  $c_1$  and  $c_2$  are two constants.

*Proof.* By (7.1), we have

$$R_{22} - R_{33} = \frac{1}{2}(A + B)(B^2 - AB + AC + BC).$$

By (1) in Lemma 7.1, we know there is time  $\tau$  such that  $CB - AB \geq 0$  for  $t \geq \tau$ . Then we get

$$R_{11} \geq R_{22} \geq R_{33}$$

and

$$\left[ \frac{1}{3}(A + B)^2 - C(A + B)^2 \right] \lambda \leq \frac{d}{dt} \lambda \leq \left[ \frac{1}{3}(A + B)^2 + A(A^2 - B^2) \right] \lambda$$

after  $\tau$ .

Then  $\lambda(t)e^{\int_\tau^t -[\frac{1}{3}(A+B)^2 - C(A+B)^2]dt}$  is nondecreasing along the normalized Ricci flow, whereas  $\lambda(t)e^{\int_\tau^t -[\frac{1}{3}(A+B)^2 + A(A^2 - B^2)]dt}$  is nonincreasing.

Furthermore, by Lemma 7.1 we estimate

$$\begin{aligned} \frac{1}{3}(A + B)^2 - C(A + B)^2 &\geq -\frac{(A_0 + B_0)^2 \left( C_0 + \frac{8}{3} \left( \frac{A_0}{B_0} \right) t \right)}{\left( 1 + \frac{8}{3}B_0^2t \right)^2} \\ &\geq -\frac{c_2}{t} \end{aligned}$$

and

$$\begin{aligned}
\frac{1}{3}(A+B)^2 + A^2(A^2 - B^2) &\leq \frac{1}{3}(A_0 + B_0) \left(1 + \frac{8}{3}B_0^2 t\right)^{-2} + (A+B)^2(A-B) \\
&\leq \frac{1}{3}(A_0 + B_0) \left(1 + \frac{8}{3}B_0^2 t\right)^{-2} \\
&\quad + 2A_0(A_0^2 - B_0^2) \left(1 + \frac{8}{3}B_0^2 t\right)^{-4} \\
&\leq c_1 t^{-2}
\end{aligned}$$

if  $t \geq \tau$ , where  $c_1$  and  $c_2$  are two constants.

Integration from  $\tau$  to  $t$  yields

$$\lambda(\tau) \left(\frac{t}{\tau}\right)^{-c_2} \leq \lambda(t) \leq \lambda(\tau) e^{c_1(\frac{1}{\tau} - \frac{1}{t})}.$$

□

## 8. E(2)

Given a metric  $g_0$ , we choose a Milnor frame such that

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = 0.$$

Under the normalization  $A_0 B_0 C_0 = 1$ , the nonzero curvature components are

$$(8.1) \quad \begin{cases} R_{11} = \frac{1}{2}A(A^2 - B^2), \\ R_{22} = \frac{1}{2}B(B^2 - A^2), \\ R_{33} = -\frac{1}{2}C(A - B)^2, \\ R = -\frac{1}{2}(A - B)^2. \end{cases}$$

Then the Ricci flow equations are

$$(8.2) \quad \begin{cases} \frac{dA}{dt} = -\frac{2}{3}A(2A + B)(A - B), \\ \frac{dB}{dt} = \frac{2}{3}B(2B + A)(A - B), \\ \frac{dC}{dt} = \frac{2}{3}C(A - B)^2. \end{cases}$$

Without loss of generality, we may assume that  $A_0 \geq B_0$ .

**Lemma 8.1.** ( Isenberg and Jackson [6] ) *Let  $(A, B, C)$  be a solution to system (8.2). Then we have the following results:*

(1)  $A \geq B$ ,  $B_0 \leq A \leq A_0$ ,  $B_0 \leq B \leq A_0$  and  $C_0 \leq C \leq C_0 \frac{A_0}{B_0}$ .

$$(2) A - B \leq (A_0 - B_0)e^{-4B_0^2 t}.$$

The flow converges to a flat metric.

**Theorem 8.1.** *Let  $\lambda(t)$  be the first eigenvalue of  $-\Delta$ . Then there is time  $\tau$  such that  $\lambda(t)e^{\int_\tau^t -[\frac{1}{3}(A-B)^2 + B(B^2 - A^2)]dt}$  is nondecreasing along the normalized Ricci flow, whereas  $\lambda(t)e^{\int_\tau^t -[\frac{1}{3}(A-B)^2 + A(A^2 - B^2)]dt}$  is nonincreasing. Moreover, we get*

$$e^{\frac{A_0^2(A_0 - B_0)}{2B_0^2}(e^{-4B_0^2 t} - e^{-4B_0^2 \tau})} \lambda(\tau) \leq \lambda(t) \leq e^{\frac{A_0^2(B_0 - A_0)}{4B_0^2}(e^{-4B_0^2 t} - e^{-4B_0^2 \tau})} \lambda(\tau)$$

for  $t \geq \tau$ .

As  $t$  goes to  $\infty$ ,  $\lambda(t)$  approaches a constant which corresponds to the eigenvalue of a flat metric.

*Proof.* It follows from (8.1) and Lemma 8.1 that

$$R_{22} - R_{33} = \frac{1}{2}(A - B)[C(A - B) - B(A + B)] \leq 0$$

after a time  $\tau$ . Then we have  $R_{11} \geq R_{33} \geq R_{22}$  and

$$\left[ \frac{1}{3}(A - B)^2 + B(B^2 - A^2) \right] \lambda \leq \frac{d}{dt} \lambda \leq \left[ \frac{1}{3}(A - B)^2 + A(A^2 - B^2) \right] \lambda$$

if  $t \geq \tau$ .

Then  $\lambda(t)e^{\int_\tau^t -[\frac{1}{3}(A-B)^2 + B(B^2 - A^2)]dt}$  is nondecreasing along the normalized Ricci flow, whereas  $\lambda(t)e^{\int_\tau^t -[\frac{1}{3}(A-B)^2 + A(A^2 - B^2)]dt}$  is nonincreasing.

More precisely, we get

$$\frac{1}{3}(A - B)^2 + B(B^2 - A^2) \geq -2A_0^2(A_0 - B_0)e^{-4B_0^2 t},$$

$$\frac{1}{3}(A - B)^2 + A(A^2 - B^2) = \frac{1}{3}(A - B)(A - B + A^2 + AB) \leq A_0^2(A_0 - B_0)e^{-4B_0^2 t} \lambda,$$

and

$$-2A_0^2(A_0 - B_0)e^{-4B_0^2 t} \lambda \leq \frac{d}{dt} \lambda(t) \leq A_0^2(A_0 - B_0)e^{-4B_0^2 t} \lambda$$

if  $t \geq \tau$ .

Then

$$-2A_0^2(A_0 - B_0)e^{-4B_0^2 t} \leq \frac{1}{\lambda} \frac{d}{dt} \lambda(t) \leq A_0^2(A_0 - B_0)e^{-4B_0^2 t}$$

which implies  $\lim_{t \rightarrow \infty} \lambda(t)$  exists.

Integration from  $\tau$  to  $t$  gives

$$e^{\frac{A_0^2(A_0 - B_0)}{2B_0^2}(e^{-4B_0^2 t} - e^{-4B_0^2 \tau})} \lambda(\tau) \leq \lambda(t) \leq e^{\frac{A_0^2(B_0 - A_0)}{4B_0^2}(e^{-4B_0^2 t} - e^{-4B_0^2 \tau})} \lambda(\tau)$$

for  $t \geq \tau$ . □

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DEPARTMENT OF APPLIED MATHEMATICS, COLLEGE OF SCIENCE, CHINA AGRICULTURAL UNIVERSITY, BEIJING, 100083, P.R. CHINA  
*E-mail address:* housb10@163.com